CONSERVATION OF ELECTROSTATIC FIELD GEOMETRY UNDER SPACE CHARGE CONDITIONS

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In [1] Tsyrlin investigated Poisson's equation in order to establish the conditions of conservation of electrostatic field geometry in the presence of a space charge ρ. It is worthwhile considering the complete system of beam equations and drawing certain conclusions about the coordinate systems in which the equipotential surfaces are given by equations $x^1 =$ = const at $\rho \neq 0$.

In the stationary case in the absence of an external magnetic field a monoenergetic regular [2] nonrelativisitic beam of charged particles with a specific charge η of the same sign and value is described by a system of equations which in the arbitrary curvilinear coordinate system x^i $(i = 1, 2, 3)$ has the form

$$g^{ik}v_iv_k = 2\varphi, \qquad e^{ikl}\partial v_i/\partial x^k = 0,$$

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(\sqrt{g}g^{ik}\frac{\partial\varphi}{\partial x^k}\right) = \rho, \quad \frac{\partial}{\partial x^i}\left(\sqrt{g}g^{ik}\rho v_k\right) = 0. \qquad (1)$$

Here v_i are the covariant velocity components, φ is the scalar potential, and ρ the space charge density. Equations (1) have been written in the dimensionless variables $r°$, $V°$, $\varphi°$, $\rho°$ (r, V are the absolute values of the radius vector and the velocity vector)

$$r = ar°, \qquad V = UV°, \qquad \varphi = -\frac{U^2}{\eta}\varphi°, \qquad \rho = \frac{U^2}{4\pi\eta a^2}\rho°$$

with the dimensionless quantity symbol omitted; a, U are constants with the dimensions of length and velocity, respectively.

The potentiality of the velocity vector makes it possible to reduce system (1) to a single fourth-order nonlinear differential equation in the action W [3, 4]:

$$\frac{\partial}{\partial x^m}\left\{g^{mn}\frac{\partial W}{\partial x^n}\frac{\partial}{\partial x^j}\left[\sqrt{g}g^{jl}\frac{\partial}{\partial x^l}\left(g^{ik}\frac{\partial W}{\partial x^i}\frac{\partial W}{\partial x^k}\right)\right]\right\} = 0,$$

$$v_i = \frac{\partial W}{\partial x^i}. \qquad (2)$$

We now pose the problem of finding in Euclidean space (see [5])

$$R^p_{\cdot rst} = 0 \qquad (3)$$

the coordinate systems x^i $(i = 1, 2, 3)$ in which $\varphi = \varphi(x^1)$. We shall restrict the discussion to two-dimensional orthogonal systems given by the equations

$$x^1 = \mathrm{Re}\, f(z), \quad x^2 = \mathrm{Im}\, f(z) \qquad (z = x + iy).$$

In this case

$$g_{11} = g_{22} = \sqrt{g}.$$

The most general form of the physical velocity components v_x^i $(i = 1, 2)$ satisfying the energy integral at $\varphi = \varphi(x^1)$ is given by the expressions

$$v_{x^1} = \sqrt{g^{11}}v_1 = \sqrt{2\varphi}\sin\vartheta, \qquad v_{x^2} = \sqrt{g^{22}}v_2 = \sqrt{2\varphi}\cos\vartheta,$$

$$\vartheta = \vartheta(x^1, x^2).$$

Equation (2) and the potential flow condition are written as follows:

$$g^{-1/4}\sin\vartheta\,(\sqrt{\varphi}\varphi'')' + [(g^{-1/4}\sin\vartheta)_1' +$$
$$+ (g^{-1/4}\cos\vartheta)_2']\sqrt{\varphi}\varphi'' = 0 \qquad (4)$$

$$g^{1/4}\cos\vartheta\,\varphi' + 2[(g^{1/4}\cos\vartheta)_1' - (g^{1/4}\sin\vartheta)_2']\varphi = 0. \qquad (5)$$

In (4), (5) a prime denotes differentiation, while the subscript denotes the coordinate with respect to which differentiation is performed.

In the case considered the conditions that must be satisfied if space (3) is to be Euclidean reduce to the single equation

$$\frac{\partial^2}{(\partial x^1)^2}\ln g + \frac{\partial^2}{(\partial x^2)^2}\ln g = 0. \qquad (6)$$

We shall assume that $\vartheta = \vartheta(x^1)$, i.e., that the v_x^ξ, like φ, depend only on x^1. Then Eqs. (4), (5) are first-order linear partial differential equations in ln g:

$$\frac{1}{4}\frac{\partial}{\partial x^1}\ln g + \frac{1}{4}\operatorname{ctg}\vartheta\frac{\partial}{\partial x^2}\ln g = (F + \ln\sin\vartheta)',$$

$$F' = [\ln(\sqrt{\varphi}\varphi'')]', \qquad (7)$$

$$-\frac{1}{2}\frac{\partial}{\partial x^1}\ln g + \frac{1}{2}\operatorname{tg}\vartheta\frac{\partial}{\partial x^2}\ln g = (f + 2\ln\cos\vartheta)', \quad f' = (\ln\varphi)' \qquad (8)$$

whose general solutions are given by the equations

$$\ln g = 4F + 4\ln\sin\vartheta + G(\xi), \qquad \xi = x^2 - \ln\sin\vartheta \qquad (9)$$

$$\ln g = -2f - 4\ln\cos\vartheta + Q(\zeta), \quad \zeta = x^2 - \ln\cos\vartheta \qquad (10)$$

Requiring that expressions (9), (10) be identical, we obtain

$$G = \beta\xi, \quad Q = \beta\zeta, \quad \beta = \mathrm{const}.$$

Thus, for ln g we have

$$\ln g = \Phi(x^1) + \beta x^2. \qquad (11)$$

It is clear that (6) is satisfied if $\Phi(x^1)$ is a linear function. Consequently,

$$g = \gamma \exp(\alpha x^1 + \beta x^2).$$

With different values of the constants α, β, γ Eq. (11) gives Cartesian x, y, polar R, ψ, and spiral q_1, q_2 coordinates [5].

We shall show that when $\vartheta = \vartheta(x^2)$ a joint solution of Eqs. (4), (5) does not exist.

In the three-dimensional case v_x^i satisfy the energy integral at $\varphi = \varphi(x_1)$, if

$$v_{x^1} = \sqrt{2\varphi}\cos\Psi\sin\vartheta, \qquad v_{x^2} = \sqrt{2\varphi}\cos\Psi\cos\vartheta,$$
$$v_{x^3} = \sqrt{2\varphi}\sin\Psi$$
$$\Psi = \Psi(x^1, x^2, x^3), \quad \vartheta = \vartheta(x^1, x^2, x^3).$$

Assuming that $\Psi = \Psi(x^1)$, $\vartheta = \vartheta(x^1)$ and requiring that Eqs. (1) transform into ordinary differential equations, we arrive at three cylindrical coordinate systems corresponding to the above-mentioned two-dimensional systems, and at the spherical coordinates, r, θ, ψ. Thus, at $\rho \neq 0$ the field geometry is preserved between parallel planes x = const, coaxial cylinders R = const, concentric spheres r = const, as well as between inclined planes ψ = const, spiral cylinders q_1 = const, q_2 = const, and cones θ = const. The first three cases, corresponding to the classical Child-Langmuir-Blodgett solutions, are well known. For these geometries one-dimensional (single-component $v_1 = dS/dx^1$, $v_2 = v_3 = 0$) flows are realized. None of the other flows enumerated have this property.

Since the solution of the beam equations is not known in advance, it evidently only makes sense to pose the problem of the conservation of field geometry when a study of this problem facilitates the finding of such a solution. It is to be expected that this will be the case only for the four coordinate systems mentioned above [6, 7], although in the present formulation it was found possible to prove weaker assertions.

REFERENCES

1. L. E. Tsyrlin, "Conditions of conservation of electrostatic field geometry under space charge conditions," ZhTF, 27, no. 7, 1957.
2. D. Gabor, "Dynamics of electron beams," Proc. IRE, vol. 33, no. 11, 1945.
3. K. Spangenberg, "Use of the action function to obtain the general differential equations of space-charge flow in more than one dimension," J. Franklin Inst., vol. 232, no. 4, 1941.
4. B. Meltzer, "Single-component stationary electron flow under space-charge conditions," J. Electr., vol. 2, no. 2, 1956.
5. V. A. Syrovoi, "Single-component beams of particles of like charge," PMTF, no. 3, 1964.
6. V. A. Syrovoi, "Group-invariant solutions of the equations of a plane stationary beam of charged particles," PMTF, no. 4, 1962.
7. V. A. Syrovoi, "Group-invariant solutions of the equations of a three-dimensional stationary beam of charged particles," PMTF, no. 3, 1963.

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